Welcome back to the second presentation in the major topic of the Time Value of Money. This module presents the effect of compounding for periods of less than one year on interest rates, payments, present values, and future values. The idea behind compounding is that interest can be earned not only on the principal but also on the periodic interest payments, if these interest payments are accumulated throughout the year. This is really what compounding means – earning interest on previous interest payments to increase the return beyond the simple interest per year. An example can easily illustrate the concept.
Suppose you have a savings account that pays interest at 12% interest per year. If you deposit $100 at the beginning of the year, you will have $112 at the end of the year when the interest of $12 is added. This represents compounding once per year and is equivalent to simple interest per year.
Let’s now consider compounding the interest every 6 months. In this case, instead of 12% interest every year, the interest rate would be 6% every 6 months to keep the interest rate per year at the same quoted annual rate.
Now, if you deposit $100 at the beginning of the year, you will have $106 at the end of 6 months.
If you leave the $106 in the account, at the end of 12 months you will have $106 times 1.06, or $112.36. Notice that this is $.36 more than the account would have had at 12% simple interest per year. Of course, this extra return is a result of compounding the interest on the interest. In this case, a 6% return on the $6 interest earned during the first 6 months yields an additional $.36 compared to 12% simple interest per year.
Notice that the calculation of the annual return in the account could be simplified to a single calculation of $100 times 1.06 squared, which equals $112.36. Of course, this represents an actual return of 12.36% per year.
Suppose the interest is compounded quarterly. In this case, instead of 12% interest every year, the interest rate would be 3% every quarter.
If you again deposit $100 at the beginning of the year, at the end of 3 months you will have $103.
At the end of 6 months, you will have $103 times 1.03, or $106.09.
At the end of 9 months, you will have $106.09 times 1.03, or $109.27.
12% / Year
Compounded Quarterly

$100 (1.03) = 103 (1.03) = 106.09 (1.03) = 109.27 (1.03) = $112.55

And at the end of one year, you will have $109.27 times 1.03, or $112.55. This represents an actual return of 12.55% per year. Notice that the effect of compounding quarterly increased the return in the account from an additional $.36 when compounding semi-annually to an additional $.55 when compounding quarterly, compared to simple interest of 12% per year.
12% / Year
Compounded Quarterly

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$100 \times (1.03) = 103 \times (1.03) = 106.09 \times (1.03) = 109.27 \times (1.03) = 112.55$

$100 \times (1.03)^4 = 112.55$

Again, the annual calculation could be accomplished in a single step by multiplying $100$ times 1.03 raised to the fourth power, which equals $112.55$. 
Now suppose the interest is compounded monthly. With monthly compounding, the interest is calculated on the remaining balance each month. In the case of the example of 12% interest per year, the interest would be 1% of the remaining balance each month.
Using the shortcut method, if you deposit $100 at the beginning of the year, at the end of the year you would have $100 times 1.01 raised to the twelfth power, which equals $112.68. This represents an actual return of 12.68% per year. Again, as we compound more times per year, the accumulated amount becomes larger and larger.
12% / Year
Compounded Continuously

$100 \ e^{0.12(1)} = \$112.75$

At the limit, when compounding takes place each instance in time, we would call this continuous time discounting. The shortcut formula for continuous discounting is $100 \times e$ raised to the $r$ times $t$ power, or 0.12, which equals $112.75$. This is the maximum compounding that can take place and represents an actual return of 12.75% per year.
At this point, we can recognize that the calculation of interest rates and returns does not depend on the amount of cash first deposited at the beginning of the year. In the case of compounding semi-annually, we can divide both sides by $100 and develop an equation with only interest rates. The actual interest rate paid is 6% every six months, and there are two six-month periods per year. Therefore, 1 plus 6% every 6 months, or 1.06, raised to the 2 power (since there are two six month periods per year), equals 1 plus the actual realized return per year. 1.06 squared equals 1.1236, which represents 12.36% actual return per year.
Likewise, 12% compounded quarterly yields 1.03 raised to the fourth power, which equals 1.1255, or an actual return of 12.55% per year.
And 12% compounded monthly yields 1.01 raised to the twelfth power, which equals 1.1268, or an actual return of 12.68% per year.
Definitions

Now that we have calculated the effects of compounding, let’s identify the specific name given to some of the interest rates we used in the example calculation.
At the beginning of the example, we quoted the interest rate as 12% per year, compounded at some frequency per year. This quote is called the APR, or Annual Percentage Return. This is a standardized way to quote annual interest rates, regardless of the compounding frequency.
For example, in the first calculation we indicated that at a quoted rate of 12% APR compounded semi-annually, what they really mean by this quote is 6% interest on the remaining balance paid every 6 months.
\[
\frac{\text{APR}}{m} = \% \text{ interest every period}
\]

\[
\text{APR} = \text{annual Percentage Return} \\
\text{m} = \text{number of periods per year}
\]

We calculate the percentage interest each period by dividing the APR by the number of compounding periods per year. This is what we did in the example problem.
12% APR
Complied Semi-Annually

\[
\frac{12\%}{2} = 6\% \text{ every 6 months}
\]

For example, 12% APR compounded semi-annually really means 6% interest on the remaining balance every 6 months.
12% APR
Compounded Quarterly

\[
\frac{12\%}{4} = 3\% \text{ every quarter}
\]

12% APR compounded quarterly really means 3% interest on the remaining balance every quarter.
And finally, 12% APR compounded monthly really means 1% interest on the remaining balance every month.
The last definition is APY, or Annual Percentage Yield. The APY is the actual realized return per year, taking into account the compounding during the year.
\[
\left( 1 + \frac{\text{APR}}{m} \right)^{m} = 1 + \text{APY}
\]

\text{APR/m = interest rate per period}
\text{m = number of periods per year}

As in our example, to calculate 1 plus the actual realized return per year (or the APY), we calculated 1 plus the return per period raised to the number of periods per year power.
\[
\left(1 + \frac{\text{APR}}{m}\right)^m = 1 + \text{APY}
\]

\[
\text{APY} = \left(1 + \frac{\text{APR}}{m}\right)^m - 1
\]

To solve for the APY directly, you simply have to subtract 1 from both sides of the equation. Therefore, the Annual Percentage Yield is equal to 1 plus the interest rate per period, raised to the m power, minus 1, where m equals the number of periods per year.
12% APR compounded semi-annually is an APY of 12.36%. The interest rate is really 6% every 6 months. Therefore, $1.06^2 - 1$ equals .1236, or 12.36% APY.
12% APR compounded quarterly really means 3% interest is paid every quarter. Therefore, $1.03^4 - 1$ equals $.1255$, or 12.55%.
12% APR
Compounded Monthly

\[
\text{APY} = \left(1 + \frac{.12}{12}\right)^{12} - 1
\]

\[
= 1.01^{12} - 1
\]

\[
= .1268
\]

\[
= 12.68\%
\]

And finally, 12% APR compounded monthly really means 1% interest is paid every month. Therefore, 1.01 raised to the twelfth power, minus 1 equals .1268, or 12.68% APY.
Example Problem

Let’s work an example problem to make sure you know how to calculate the Annual Percentage Yield from the APR.
Suppose the bank quotes a savings account with a 6% APR interest rate compounded quarterly. What is the Annual Percentage Yield for this account?
6% APR
Compounded Quarterly

$$\text{APY} = \left(1 + \frac{0.06}{4}\right)^4 - 1$$

$$= 1.015^4 - 1$$

$$= 0.0614$$

$$= 6.14\%$$

6% compounded quarterly really means 6% divided by 4, or 1.5% interest is paid on the remaining balance each quarter. Therefore, 1.015 raised to the fourth power, minus 1 equals .0614, or 6.14% APY.
Loans are quoted on an APR basis. The compounding period can be determined by the frequency of the payments.

Loans are quoted on an APR basis the same way as a savings account. Sometimes the compounding period is not explicitly stated. However, the compounding period can be determined by the frequency of the payments. This may be a little clearer with another example.
Suppose a bank quotes a mortgage loan on a house at 7.2% APR.
Since you know you will be making monthly payments on the loan, the compounding frequency must be monthly also. What would be the Annual Percentage Yield for a 7.2% APR home mortgage?
7.2% APR with monthly payments really means 0.6% interest on the remaining balance each month. Therefore, 1.006 raised to the twelfth power, minus 1 equals 0.0744, or 7.44% APY.

Let’s do one more example, and then move on.
Suppose an automobile distributorship offers 2.4% APR financing for a new vehicle.
Since you intend to make monthly payments, you realize this means 2.4% APR compounded monthly. What would be the Annual Percentage Yield for a 2.4% APR vehicle loan?
2.4% APR with monthly payments really means 0.2% interest on the remaining balance each month. Therefore, 1.002 raised to the twelfth power, minus 1 equals .0243, or 2.43% APY,
PV and FV for compounding periods less than one year

The next part of this presentation applies the same concepts and equations for Present Values and Future Values from the previous module, but now for compounding periods of less than one year.
Recall from the previous module the equations we used for Present Values and Future Values. For a single cash flow, the Future Value is equal to the Present Value times \((1+r)^t\) raised to the \(t\) power. And the Present Value is equal to the Future Value divided by \((1+r)^t\) raised to the \(t\) power. In both of these cases, \(r\) was the interest rate per year and \(t\) was the number of years.
With compounding for periods of less than 1 year, the same formulas apply, but the concept of $r$ and $t$ is somewhat different in that they are not constrained to 1-year periods. For example, $r$ now equals the interest rate per period, and $t$ equals the number of periods. Other than this change in concept, the formulas work the same as before.
Formulas for annuities:

\[
FVA = C \left[ \frac{(1+r)^t - 1}{r} \right]
\]

\[
PVA = C \left[ 1 - \frac{1}{(1+r)^t} \right]
\]

\( r \) = interest rate per year  
\( t \) = number of years  
\( C \) = payment per year

Also recall from the previous module the equations we used for the Present Value of an Annuity and the Future Value of an Annuity. Again, the interest rate per year, the number of years, and the payment per year were all on an annual basis.
Formulas for annuities:

\[ FVA = C \left( \frac{(1+r)^t - 1}{r} \right) \]
\[ PVA = C \left( 1 - \frac{1}{(1+r)^t} \right) \frac{1}{r} \]

- \( r \) = interest rate per period
- \( t \) = number of periods
- \( C \) = payment per period

For compounding with periods less than one year, these formulas work the same way as before. The only change necessary is that these three items need to be converted to a compounding period basis rather than years.

A few example problems will clarify how to apply the Future Value and Present Value equations for a single cash flow or for an annuity, when there is compounding for periods of less than one year.
Suppose you deposit $600 in a savings account that is quoted at 6% APR compounded quarterly. How much money will be in this account if you leave all the funds in this account for 5 years?
First, we need to recognize that the problem is asking for the Future Value of a single cash flow. So we start with the equation, the Future Value equals the Present Value times \((1+r)^t\) raised to the \(t\) power.
The next step is to identify the data we need for the rest of the equation. In this case, we need the Present Value, r, and t. The idea for compounding with periods less than one year is to convert the input data from years to compounding periods. For this problem, compounding is quarterly, so the input data must be converted to a quarterly basis.
\[ PV = 600 \]
\[ r = \]
\[ t = \]
\[ FV = PV (1+r)^t \]

The Present Value in this problem is $600 and needs no conversion.
PV = 600
r = 6%/4 = 1.5% per quarter
t =
FV = PV (1+r)^t

The interest rate is quoted as 6% APR, which is on an annual basis. 6% per year divided by 4 quarters per year equals 1.5% per quarter.
PV = 600
r = 6%/4 = 1.5% per quarter
\( t = 5 \times 4 = 20 \) quarters
\[ FV = PV \times (1+r)^t \]

Likewise, the number of years in the account must be converted to a quarterly basis. 5 years times 4 quarters per year equals 20 quarters.
PV = 600
r = 6%/4 = 1.5% per quarter
t = 5 (4) = 20 quarters

FV = PV (1+r)^t
    = 600 (1.015)^{20}
    = 808.11

Plugging the numbers into the equation, 600 times 1.015 raised to the twentieth power equals a Future Value of $808.11.

Let’s look at another example problem
Need $8,000 when niece is 18
She is currently 8 years old
6.5% APR
Compounded Weekly
Deposit how much today?

Suppose that you have a very close relationship with your niece, and you would like to set up a trust fund so that she could buy a modest used car (perhaps about $8,000 for a decent car) when she reaches the age of 18. She is currently 8 years old. Your Credit Union is offering a special long-term savings account quoted at 6.5% APR, compounded weekly. How much money do you need to place in the trust fund today so that the trust fund will be worth $8,000 when your niece is 18?
PV = FV / (1+r)^t

To solve this problem, we first need to recognize that we want to calculate the Present Value of a single cash flow. So we start with the equation, the Present Value equals the Future Value divided by (1+r) raised to the t power.
The next step is to identify the input data we need for the rest of the equation. In this case, we need the Future Value, \( r \), and \( t \). Since the compounding period is weekly, we need to convert the data to a weekly basis.
The Future Value in this problem is $8,000 and needs no conversion.
FV = 8,000
r = 6.5%/52 = .125% per week
t =

PV = FV / (1+r)^t

The interest rate is quoted as 6.5% APR, which needs to be converted to a weekly basis. 6.5% divided by 52 weeks in a year equals .125% per week.
\[ FV = 8,000 \]
\[ r = \frac{6.5\%}{52} = .125\% \text{ per week} \]
\[ t = 10 \times 52 = 520 \text{ weeks} \]

\[ PV = \frac{FV}{(1+r)^t} \]

\( t \) is equal to 10 years in this problem and needs to be converted to weeks. 10 years times 52 weeks in a year equals 520 weeks.
\[ FV = 8,000 \]
\[ r = \frac{6.5\%}{52} = .125\% \text{ per week} \]
\[ t = 10 \times 52 = 520 \text{ weeks} \]

\[ PV = \frac{FV}{(1+r)^t} \]
\[ = \frac{8,000}{(1.00125)^{520}} \]
\[ = 4,178.06 \]

Plug in the numbers into the equation, 8000 divided by 1.00125 raised to the 520 power equals $4,178.06.

Let’s look at another example problem.
A financial planner recently advised you to start saving for your retirement right now, even though you are only 25 years old. He also advised you to place your savings regularly into a stock mutual fund, which he conservatively estimates will earn about 8.4% APR compounded monthly. You decide this is a good idea and plan to regularly deposit $100 per month into a stock mutual fund. At the assumed rate, how much will you accumulate in the mutual fund account at age 65?
First, we need to recognize that the problem uses an annuity. It is an annuity because you plan to deposit the same amount of money each and every month until you are 65 years old. Next, the problem asks how much you will accumulate by age 65, which is 40 years in the future. From this information, you should recognize that the solution to this problem would require the Future Value of an Annuity formula. So we start with the equation, the Future Value of an Annuity is equal to $C$, the constant payment each period, times a Future Value of an Annuity factor. This factor is $(1+r)^t$ raised to the $t$ power, minus 1, divided by $r$. 

$$FVA = C \left[ \frac{(1+r)^t - 1}{r} \right]$$
To complete the calculation, we will need C, r, and t. Since there is compounding on a monthly basis, we need to convert this input data to a monthly basis.
C, the constant payment each period, is given in the problem as $100 per month.
C = 100 per month
r = 8.4%/12 = 0.7% per month
t = 
FVA = C \left[ \frac{(1+r)^t-1}{r} \right]

r is given in the problem as 8.4% APR compounded monthly. On a monthly basis, this is 8.4% per year divided by 12 months in a year, which equals 0.7% interest per month.
\[ C = 100 \text{ per month} \\
\[ r = \frac{8.4\%}{12} = 0.7\% \text{ per month} \\
\[ t = 40 (12) = 480 \text{ months} \\
\]

\[ FVA = C \left[ \frac{(1+r)^t - 1}{r} \right] \]

t is given in the problem as 40 years (age 65 minus age 25). 40 years times 12 months per year equals 480 months.
C = 100 per month  
r = 8.4%/12 = 0.7% per month  
t = 40 (12) = 480 months

\[ FVA = C \left( \frac{(1+r)^t - 1}{r} \right) \]

\[ FVA = 100 \left[ \frac{(1.007)^{480} - 1}{.007} \right] = 100 \left[ \frac{27.454}{.007} \right] \]

\[ = 100 \left[ 3922.02 \right] = 392,202 \]

Recall from the previous module that the best way to evaluate an annuity formula is to start with 
(1+r) and work your way outwards in the equation. 1.007 raised to the 480 power, minus one, 
equals the numerator of 27.454, divided by .007 equals the Future Value of an Annuity factor 
of 3,922.02, times 100 equals $392,202.

Let’s look at one more example problem.
Suppose you are interested in buying a new car for $25,000. The dealership has offered you 3.6% APR financing with nothing down on a 5-year loan. What would be the monthly payment for purchasing this car?
The first step in solving this problem is to recognize that the problem involves an annuity, since you plan to make a constant payment each and every month over the next 5 years. But should you use the Future Value of an Annuity equation or the Present Value of an Annuity equation?

The problem indicates that the dealership would be willing to accept a constant payment from you each and every month in place of the $25,000 in cash today. This $25,000 today should lead you to the correct choice in an annuity equation.
This problem requires the Present Value of an Annuity equation because the $25,000 is a Present Value. The formula for the Present Value of an Annuity is \( C \), the constant payment each period, times the Present Value of an Annuity factor. This factor is one minus one over \((1+r)^t\) raised to the \(t\) power, divided by \(r\).
Since we want to know the constant payment each month, the input data we need to solve this problem is the Present Value of the Annuity, \( r \), and \( t \).

As before, we need to convert the input data to the compounding period basis. However, the problem statement did not explicitly identify the compounding period. To determine the compounding period, we simply need to recognize that the compounding period is usually equal to how often the payments are made. In this problem, the payments are made monthly. So the financing implicitly must be 3.6% APR compounded monthly. We therefore need to convert all the input data to a monthly basis.
The Present Value of the Annuity is given as $25,000.
The interest rate is quoted as 3.6% APR, which is equal to 3.6% divided by 12, or .3% per month.
PVA = 25,000
r = 3.6%/12 = 0.3% per month
t = 5 (12) = 60 months

\[ PVA = C \left( 1 - \frac{1}{(1+r)^t} \right) \]

\( t \) is equal to 5 years times 12 months per year, or 60 months.
PVA = 25,000
r = 3.6%/12 = 0.3% per month
t = 5 (12) = 60 months

\[ PVA = C \left[ \frac{1 - \frac{1}{(1+r)^t}}{r} \right] \]

\[ 25,000 = C \left[ 1 - \frac{1}{(1.003)^{60}} \right] = C \left[ \frac{.1645}{.003} \right] = C \left[ 54.835 \right] \]

To solve the equation for C, on a calculator we begin with the \((1+r)\). 1.003 raised to the 60th power equals 1.1969. One over that (pressing the 1/x key) is .8355. Changing the sign plus one equals the numerator of .1645. Dividing by .003 yields the Present Value of Annuity factor of 54.835.
PVA = 25,000
r = 3.6%/12 = 0.3% per month
t = 5 (12) = 60 months

\[
PVA = C \left[ \frac{1 - \left( 1 + \frac{1}{1+r} \right)^t}{r} \right]
\]

\[
25,000 = C \left[ \frac{1 - \left( 1.003 \right)^{60}}{.003} \right] = C \left[ \frac{1645}{.003} \right] = C \left[ 54.835 \right]
\]

C = 455.91

To solve for C, you need to press the 1/x key and multiply times $25,000. This yields a monthly payment of $455.91.

Hopefully, with these example problems you can see how to apply the four basic time value of money equations to solve a multitude of problems, including those which use compounding for periods less than one year.
Summary

Let’s summarize what we have covered in this module.
Summary

• APR Converted to APY

First, we defined the APR method of quoting annual interest rates and how to convert such quotes to an APY basis, which is equivalent to a simple interest per year.
Summary

• APR Converted to APY
• Applied the FV, PV, FVA, and PVA equations

Then we showed that the four basic time value of money equations work the same way as before, except they can be applied to compounding periods rather than just years.
Summary

- APR Converted to APY
- Applied the FV, PV, FVA, and PVA equations
- Illustrated with example problems

And finally, we illustrated the application of compounding for periods less than one year with several example problems.

This completes the Time Value of Money module on compounding with periods less than one year.